II. "Invariants, Covariants, and Quotient Derivatives associated with Linear Differential Equations." By A. R. Forsyth, M.A., F.R.S., Fellow of Trinity College, Cambridge. Received January 7, 1888.

(Abstract.)

The present memoir deals with the covariantive forms associated with the general ordinary linear differential equation; it is strictly limited to the consideration of those forms, without any discussion of their critical character.

The most general transformation, to which such equation can be subjected without change of linearity or of order, is one whereby the dependent variable y is transformed to u by a relation

$$y = uf(x),$$

and, at the same time, the independent variable x is changed to z; and, when these transformations are effected on

$$\sum_{r=0}^{r=n} \mathrm{P}_r \frac{n!}{r!\,n-r!} \frac{d^{n-r}y}{dx^{n-r}} = 0,$$

so that it becomes

$$\sum_{r=0}^{r=n} \mathrm{Q}_r \frac{n!}{r!\,n-r!} \frac{d^{n-r}u}{dz^{n-r}} = 0,$$

there are r relations between the coefficients P and Q; and P_0 and Q_0 may manifestly, without loss of generality, be taken equal to unity.

It is shown that, from these relations, others can be deduced which are of the type

$$\psi(\mathrm{P}) = \left(\frac{dz}{dx}\right)^\rho \psi(\mathrm{Q}),$$

where $\psi(\mathrm{P})$ is an algebraical function of the coefficient P and their derivatives. Such a function is called an invariant of index ρ; and irreducible invariants are of two classes, fundamental and derived.

It is convenient to have expressions for the invariants, when the differential equation has an implicitly general canonical form. In the first place it may be supposed that P_1 and Q_1 are both zero; otherwise both equations can by substitutions of Y for $ye^{\int \mathrm{P}_1 dx}$ and U for $ue^{\int \mathrm{Q}_1 dz}$ be reduced to forms in which the terms involving the $(n-1)$th differential coefficients of the dependent variable do not occur. The relation between the dependent variables is now

$$y = uz'^{-\frac{1}{2}(n-1)};$$

and the expressions of the simplest invariants are

$$\Theta_3 = P_3 - \frac{3}{2} P_2',$$

$$\Theta_4 = P_4 - 2P_3' + \frac{6}{5} P_2'' - \frac{3}{5} \frac{5n+7}{n+1} P_2^2,$$

$$\Theta_5 = P_5 - \frac{5}{2} P_4' + \frac{15}{7} P_3'' - \frac{5}{7} P_2''' - \frac{10}{7} \frac{7n+13}{n+1} P_2 \Theta_3,$$

similar expressions being obtained for Θ_6 and Θ_7. The expression for each of the $n-2$ invariants of this class is shown to consist of two parts, one of which is linear in the coefficients and their derivatives, the other of which is not linear but every term contains as a factor either P_2 or some derivative of P_2.

It is then proved that there is an implicitly general form of the equation for which both Q_1 and Q_2 vanish; this form, taken as the canonical form, is obtainable (as is known from earlier investigations) by the previous determination of the multiplier of the dependent variable and by the determination of the independent variable from the equation

$$\{z, x\} = \frac{6}{n+1} P_2,$$

or its equivalent, $$\frac{d^2\theta}{dx^2} + \frac{3}{n+1} P_2 \theta = 0,$$

where $z' = \theta^{-2}$.

For this canonical form of equation the expressions of the foregoing $n-2$ invariants are given in the form

$$\Theta_\sigma = Q_\sigma + \tfrac{1}{2}\sigma \sum_{r=1}^{r=\sigma-3} (-1)^r \alpha_{r,\sigma} \frac{d^r Q_{\sigma-r}}{dz^r},$$

where $\alpha_{1,\sigma}$ is unity and, for values of r greater than 1,

$$\alpha_{r,\sigma} = \frac{(\sigma-1)(\sigma-2)^2(\sigma-3)^2 \ldots (\sigma-r+1)^2(\sigma-r)}{2 \cdot 3 \ldots r (2\sigma-3)(2\sigma-4) \ldots (2\sigma-r-1)}.$$

These invariants are called *priminvariants.*

The proof of these results occupies the second section of the memoir. The first section is devoted to a short historical sketch of the growth of the subject, reference being made to the investigations of Cockle, Laguerre, Brioschi, Malet, and, especially, of Halphen, all of whom have, so far as concerns the theory of forms, discussed either seminvariants only or, with the single exception of Θ_3 for the $n$tic, in-

variants of the cubic and the quartic in forms which differ from the canonical form herein adopted.

In the third section *derived invariants* are obtained, all in their canonical forms; they are derived from the priminvariants by one or other of two processes called the quadriderivative and the Jacobian. The irreducible invariants are ranged in classes according to their degrees. The quadrinvariants consist of $n-2$ functions,

$$\Theta_{\sigma,1} = 2\sigma\Theta_\sigma\Theta_\sigma'' - (2\sigma+1)\Theta_\sigma'^2,$$

and of $n-3$ independent functions of the form

$$\lambda\Theta_\lambda\Theta_\mu' - \mu\Theta_\mu\Theta_\lambda';$$

and every class of invariants of degree higher than the second contains $n-2$ invariants, each in that class associated with one of the priminvariants in successive derivation according to the law

$$\Theta_{\sigma,r} = \sigma\Theta_\sigma\Theta'_{\sigma,r-1} - r(\sigma+1)\Theta_\sigma'\Theta_{\sigma,r-1}.$$

Propositions relating to the dependence of the derived invariants are proved in the section; and simpler equivalent forms are obtained later in the memoir.

In the fourth section covariants are discussed. The transformation of the dependent variable in the second section shows that, with the adopted definition of invariance, viz., reproduction save as to a power of z', the dependent variable is a covariant. A set of dependent variables, associate with the original dependent variable, is obtained by the application of a theorem due to Clebsch. Denoting these by v_2, v_3, ..., v_{n-1}, for the untransformed equation, and by t_2, t_3, ..., t_{n-1}, for the transformed equation, we have

$$v_p = t_p z'^{-\frac{1}{2}p(n-p)},$$

so that these associate variables are covariants. The variable v_p satisfies a linear differential equation of order $\dfrac{n\,!}{p\,!\,\overline{n-p}\,!}$; and, in particular, v_{n-1} is the variable of Lagrange's "adjoint" equation. The following inferences relating to these variables and equations are made:—

(α) The dependent variables form a complete system, that is, functional combinations of them, similar to those by which they are obtained, are expressible in terms of members of the system;

(β) The associate linear equations in variables which have the same index are mutually adjoint;

(γ) The invariants of the associate linear equations are expressible in terms of the invariants of the original equation.

In the fifth section these dependent variables are treated in the same manner as the priminvariants in the third, and give two classes of functions—*identical covariants*, which in their canonical form involve dependent variables only, and *mixed covariants*, which involve dependent variables and coefficients of the original equation. The former class includes series of covariants, each involving only one of the dependent variables; the law of successive formation is

$$\mathrm{V}_{p,1} = p(n-p)v_p v_p'' - (np-p^2-1)v_p'^2,$$
$$\mathrm{V}_{p,r+1} = p(n-p)v_p \mathrm{V}'_{p,r} - r(np-p^2-2)\mathrm{V}_{p,r}\, v_p',$$

for each of the associate variables. But other functions which involve more than one of the variables, *e.g.*, the Jacobian of two of them, are omitted, for they can be algebraically compounded by means of the mixed covariants. The number of independent identical covariants in the succession is one less than the order of the equation satisfied by the variable: but a modification of this number is necessary when they are considered as covariants of a differential quantic instead of being considered covariants of a differential equation. For in this case we must either retain the quantic and all derivatives from it—when there is no modification of the number of identical covariants; or the number is unlimited, and then the quantic and its derivatives are composite.

The mixed covariants which are irreducible are proved to consist only of first Jacobians of some one of the invariants and all the dependent variables in turn.

The aggregate of the concomitants is constituted by the three classes of functions thus obtained, viz., invariants, identical covariants, and mixed covariants.

In the sixth section the results previously derived are applied to equations of the second, the third, and the fourth orders; solely, however, for the sake of illustration and not for purposes of critical discussion of classes of these equations.

For the equation of the second order the only result obtained is a reproduction of Schwarz's theorem; the equation has no invariant.

For the equation of the third order, the canonical form of which is

$$u''' + \Theta_3 u = 0,$$

and which has a single priminvariant, one or two questions are solved; in particular, the differential equation satisfied by the quotient of two solutions of the cubic is obtained, and there is thence deduced a quotient-derivative, which is the analogue of Schwarz's derivative for the quadratic.

For the equation of the fourth order there are two canonical forms, viz.:—

$$u^{\text{iv}}+4Q_3u'+Q_4u = 0,$$
$$u^{\text{iv}}+6R_2u''+R_4u = 0,$$

to which the explicitly general quartic can be reduced by the solution of linear differential equations of the second and the third order respectively. The differential equation satisfied by the quotient of two solutions of the quartic is obtained; and in this connexion there arises a quartic quotient-derivative. Finally, the associate equations of the quartic are formed; and it is verified that all their invariants are expressible in terms of the invariants of the original quartic.

The seventh section is really a digression from the main subject of the paper; it is concerned with the special class of functions which occur in the preceding section and are called quotient-derivatives. The quotient-derivatives of lowest order are

$$\begin{vmatrix} s'' , & 2s' \\ s''' , & 3s'' \end{vmatrix} = [s, z]_2;$$

$$\begin{vmatrix} s''' , & 3s'' , & 3s' \\ s^{\text{iv}} , & 4s''' , & 6s'' \\ s^{\text{v}} , & 5s^{\text{iv}} , & 10s''' \end{vmatrix} = [s, z]_3;$$

and so on; in these the differential coefficient of highest order which occurs is of odd order, and thence these derivatives are said to be *of odd order*. The two most important propositions which relate to them are, first, if

$$[\sigma, s]_m = 0, \qquad [s, z]_n = 0, \qquad [z, x]_p = 0,$$

then
$$[\sigma, x]_\rho = 0,$$

where
$$\rho-1 = (m-1)(n-1)(p-1);$$

and second, that the law of change for homographic transformation of both variables is

$$\left[\frac{as+b}{cs+d}, \frac{ez+f}{gz+h}\right]_n = \frac{(ad-bc)^n}{(eh-fg)^{n^2}} \frac{(gz+h)^{2n^2}}{(cs+d)^{2n}} [s, z]_n.$$

There is then investigated the series of similar functions *of even order* in the form

$$\begin{vmatrix} s' , & s \\ s'' , & 2s' \end{vmatrix} , \qquad \begin{vmatrix} s'' , & 2s' , & s \\ s''' , & 3s'' , & 3s' \\ s^{\text{iv}} , & 4s''' , & 6s'' \end{vmatrix}$$

and so on; and a connexion between the two classes is given.

Up to this point the results in the memoir which relate to the derivation of covariantive forms have been synthetically obtained; the eighth (and last) section relates to their analytical derivation. It is shown that, for a homographic transformation of the independent variable applied concurrently with the proper transformation of the dependent variable, the canonical form of the differential equation is maintained. These transformations are applied to prove, by the method of infinitesimal variation, that every concomitant ϕ in its canonical form satisfied the linear partial differential equation

$$\sum_{m-1}^{m=n-1}\left\{m(n-m)u^{(m-)}\frac{d\phi}{du^{(m)}}\right\}$$

$$+\sum_{p=2}^{p=n-1}\sum_{r=1}^{r=\sigma-1}\left[r\{p(n-p)-r+1\}v_p^{(r-1)}\frac{d\phi}{dv_p^{(r)}}\right]$$

$$=\sum_{\mu=3}^{\mu=n}\sum_{s=1}\left\{s(2\mu+s-1)\Theta_\mu^{(s-1)}\frac{d\phi}{d\Theta_\mu^{(s)}}\right\},$$

where $\sigma=\dfrac{n\,!}{p\,!\,n-p\,!}$. This is called the *form-equation.* Such a concomitant ϕ also satisfies the equation

$$\sum_{m=0}^{m=n-1}\left[\left\{m-\tfrac{1}{2}(n-1)\right\}u^{(m)}\frac{d\phi}{du^{(m)}}\right]$$

$$+\sum_{p=2}^{p=n-1}\sum_{r=0}^{r=\sigma-1}\left[\{r-\tfrac{1}{2}p(n-p)\}v_p^{(r)}\frac{d\phi}{dv_p^{(r)}}\right]$$

$$=\lambda\phi-\sum_{\mu=3}^{\mu=n}\sum_{s=0}\left\{(s+\mu)\Theta_\mu^{(s)}\frac{d\phi}{d\Theta_\mu^{(s)}}\right\},$$

where λ is the index of the concomitant. This is called the *index equation;* and, when the form of ϕ is known, it merely determines λ, which can be written down from an inspection of the concomitant.

These equations are applied, (i) to the identical covariants in u,—(ii) to the invariants derived from Θ_3,—for each of which simplified equivalent functions are obtained for derivatives of order higher than the third,—and (iii) to verify that the Jacobian of a priminvariant and any of its derived invariants satisfies the equations. Lastly, by means of the theory of partial differential equations, it is proved that the aggregate of concomitants obtained in the earlier part of the memoir is complete, that is, that any concomitant can be expressed as an algebraical function of the members of that aggregate.